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Operators with Closed Ranges in Spaces of Analytic Vector-Valued Functions

ZBIGNIEW SLODKOWSKI*

Department of Mathematics,
University of California, Los Angeles, California 90024

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Let $A_+(P, X)$ denote the Banach space of X-valued analytic functions on a polydisc $P \subset \mathbb{C}^n$ with absolutely convergent Taylor series. *Main result*: Let $T(z)$ be an analytic family of bounded operators from X to Y. Assume that all $T(z)$ have closed ranges depending continuously on z. Then the "multiplication" operator $\tilde{T}: A_+(P, X) \to A_+(P, Y)$, induced by $T(z)$, has closed range. (Equivalent characterization of such operator-valued functions are given.) This result makes it possible to construct Banach spaces of sections of some infinite-dimensional analytic sheaves. The construction is functorial and has certain exactness properties which help to study analytic perturbations of the joint spectrum of J. L. Taylor.

INTRODUCTION

The following general problem is the motivation of this paper: having given an exact sequence of analytic sheaves

$$0 \to F^0 \to F^1 \to \cdots \to F^n \to 0 \tag{1}$$

satisfying some natural conditions construct a corresponding exact sequence of Banach spaces of sections of these sheaves (and do this over sufficiently rich family of subsets of the base space). It does not seem that this problem was studied enough, as compared with the similar one concerning Frechet spaces of sections; cf. Taylor [8, 9], Leiterer [4]. In Sections 3 and 5 we solve two instances of the general problem, encountered by the author during his work on analytic perturbation of Taylor spectrum [7]. To get these results, a detailed study of analytic families of operators with closed ranges is needed, to which most of the paper is devoted.

* Present address: Department of Mathematics, Statistics, and Computer Science, The University of Illinois at Chicago, Chicago, Illinois 60680.

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In the basic example of the general problem mentioned above, F^i, $0 \leqslant i \leqslant m$, is the sheaf of germs of analytic X^i-valued functions over $G \subset \mathbb{C}^k$, where X^i are complex Banach spaces and the sheaf homomorphisms are operators of multiplication by analytic, operator-valued functions $z \to d^i(z)$: $G \to L(X^i, X^{i+1})$, $0 \leqslant i \leqslant m-1$. Such sequence of sheaf homomorphisms is exact if and only if $\operatorname{Im} d^i(z) = \ker d^{i+1}(z)$, $z \in G$, $-1 \leqslant i \leqslant m$ ($d^m = 0 = d^{-1}$); cf. Taylor [8]. Under these assumptions we prove in Section 3 that

if P is the polydisc $D(a_1, r_1) \times \cdots \times D(a_k, r_k)$, $r_i > 0$, such that $\bar{P} \subset G$, then there exist Banach spaces of analytic X^i-valued functions on P, namely $A_+(P, X^i)$ (to be defined next) such that the operator of multiplication by $d^i(z)$ maps $A_+(P, X^i)$ into $A_+(P, X^{i+1})$ and the complex thus defined

$$0 \to A_+(P, X^0) \to A_+(P, X^1) \to \cdots \to A_+(P, X^m) \to 0$$

is exact.

(Note that if all operators $d^i(z)$ have generalized inverses, then there are many other ways of constructing Banach spaces of sections).

The space $A_+(P, X)$ needed above is the Banach space of all functions $f: P \to X$ such that

$$f(z) = \sum_{i_1, \ldots, i_k \geqslant 0} x_{i_1, \ldots, i_k} z_1^{i_1} \cdots z_k^{i_k},\ z \in P \quad \text{and} \quad \sum_{i_1, \ldots, i_k} \|x_{i_1, \ldots, i_k}\| \, r_1^{i_1} \cdots r_k^{i_k} < \infty. \tag{2}$$

The above result follows immediately from the next theorem. (Below, $H(G, X)$ denotes the space of all X-valued analytic functions on G.)

THEOREM 1. *Let P be a compact polydisc $P \subset G \subset \mathbb{C}^k$, and $T \in H(G, L(X, Y))$, where X and Y are complex Banach spaces. Assume that there is a Banach space F and a function $B \in H(G, L(Y, F))$ such that $\operatorname{Im} T(z) = \ker B(z)$ for all $z \in P$. Then the "multiplication" operator $\tilde{T}: A_+(P, X) \to A_+(P, Y)$ has closed range. More specifically*

$$\operatorname{Im} \tilde{T} = \{ g \in A_+(P, Y) : g(z) \in \operatorname{Im} T(z),\ z \in P \}. \tag{3}$$

The next theorem gives several characterizations of those operator-valued functions $T(z)$ to which the last result applies. In the proof of both theorems an important role is played by the reduced minimum modulus $\gamma(T)$, which characterizes quantitatively operators with closed ranges. We will use more frequently its reciprocal

$$k(T) = \sup_{y \in \operatorname{Im} T,\ \|y\| \leqslant 1} \ \inf_{Tx = y} \|x\|.$$

THEOREM 2. *Let X and Y be complex Banach spaces and M be a Stein manifold. Let $z \to T_z: M \to L(X, Y)$ be an analytic function. Assume that all operators T_z have closed ranges. Then the following conditions are equivalent*:

(i) *the function $z \to k(T_z)$ is locally uniformly bounded* (*or equivalently: continuous*);

(ii) *the set $\{(z, y): z \in M, y \in \operatorname{Im} T_z\}$ is closed in $M \times Y$*;

(iii) *the function $z \to T_z$ satisfies formal solvability condition* (FSC) *at each point $a \in M$*;

(iv) *there is a Banach space E and a function $A \in H(M, L(X, Y))$ such that the function $k(A(\cdot))$ is continuous and* $\ker T(z) = \operatorname{Im} A(z)$ *for every $z \in M$*;

(v) *there is a Banach space F and a function $B \in H(M, L(Y, F))$, such that* $\operatorname{Im} T(z) = \ker B(z)$ *for every $z \in M$*;

(vi) *there is F and $B(\cdot)$ satisfying* (v) *and such that* $\operatorname{Im} B(z)$ *is closed for all $z \in M$ and the function $k(B(\cdot))$ is continuous.*

The FSC mentioned in (iii) is a purely algebraic relation among operators that are coefficients of the power series representing $T(z)$ with respect to some coordinate system at $z = a$. The condition plays the crucial role in the proofs of both theorems, which seems to be due to the following special feature of FSC. In spite of the fact that the property of an operator having closed range is not stable with respect to analytic perturbations, it is still stable with respect to those perturbations $T(z)$ which satisfy FSC at $z = a$ (cf. Corollary 2.5, 4.2).

The paper is organized as follows: Section 1 collects for easy reference some properties and estimates of the function $k(T)$. In Section 2 FSC is used to get structure result for linear pencils of operators with closed ranges (Theorem 2.1) which is used in Section 3 to prove Theorem 1. Section 4 is devoted to the proof of Theorem 2. In Section 5 Theorems 1 and 2 are combined to help obtain exact sequences of Banach spaces of selections of some quotient sheaves, which constitutes another instance of the general problem discussed at the beginning of the Introduction.

1. PROPERTIES OF THE REDUCED MINIMUM MODULUS

Terminology

To shorten the wording of many results we make some conventions. All Banach spaces in this paper are assumed to be complex. The space of all bounded operators from X to Y is denoted by $L(X, Y)$ only if X and Y are Banach spaces; thus the expression "$T \in L(X, Y)$" implies automatically that X and Y are complex Banach spaces. We say that $T \in L(X, Y)$ is an embedding if $\operatorname{Im} T$ (= the range of T) is closed and T has zero kernel. Closure is denoted by Cl.

We begin with some properties of $k(T)$ that are obvious or well known.

Remark 1.1. (i) If $T \in L(X, Y)$, then T has closed range if and only if $k(T) < \infty$.

(ii) Let $T \in L(X, Y)$ and $M > 0$ be given. Assume that for every $y \in \operatorname{Im} T$ such that $\|y\| < 1$, there is $x \in X$ such that $Tx = y$ and $\operatorname{dist}(x, \ker T) \leqslant M$. Then $k(T) \leqslant M$.

The following modification of this criterion is implicit in Taylor [8, Proof of Lemma 2.1].

LEMMA 1.2. *Let $T \in L(X, Y)$, $M > 0$ and $0 < \delta < 1$. Assume that for every $y \in \operatorname{Im} T$ with $\|y\| < 1$ there is $x \in X$ such that $\|Tx - y\| \leqslant \delta$ and $\operatorname{dist}(x, \ker T) \leqslant M$. Then $k(T) \leqslant M/(1 - \delta)$.*

LEMMA 1.3 (Leiterer [3, Lemma 6]). *Let $T_n \in L(X, Y)$, $n = 1, 2, \ldots$. Assume that $\sup_n k(T_n) < \infty$ and $\lim_n \|T_n - T\| = 0$ for some $T \in L(X, Y)$. Then $\operatorname{Im} T$ is closed and $k(T) = \lim_n k(T_n)$.*

THEOREM 1.4 (Markus, cf. Leiterer [3, pp. 115–116]. *Let Γ be a metric space and let $T: \Gamma \to L(X, Y)$ be a continuous function. Then the following conditions are equivalent*:

(i) *the function $k(T(\cdot))$ is locally bounded on Γ*;

(ii) *the function $k(T(\cdot))$ is continuous on Γ*;

(iii) *the correspondence $z \to \operatorname{Im} T(z)$, $z \in \Gamma$, is continuous in the sense of the gap topology*;

(iv) *the correspondence $z \to \ker T(z)$, $z \in \Gamma$, is continuous in the sense of the gap topology.*

COROLLARY 1.5. *Let $T_n \in L(X, Y)$ be embeddings. Assume that $\lim_n \|T_n - T\| = 0$, and $\sup_n k(T_n) < \infty$. Then T is an embedding as well.*

Proof. By Lemma 1.3 and Theorem 1.4(iv). Q.E.D.

Remark 1.6. Let $T \in L(X, Y)$. Then $k(T) = k(T^*)$.

PROPOSITION 1.7. *Let $T \in L(X, Y)$, $S \in L(Y, Z)$. Assume that $S \circ T = 0$. Then $\operatorname{Im} T = \ker S$ and $\operatorname{Im} S$ is closed if and only if $\operatorname{Im} S^* = \ker T^*$ and $\operatorname{Im} T^*$ is closed.*

Proof. Application of Hahn–Banach theorem and of the open mapping theorem. Q.E.D.

In the remaining part of this section we sketch proofs, rather routine, of some estimates and properties of $k(T)$, needed below.

LEMMA 1.8. *Let $T \in L(X, Y)$ and let $X_0 \subset X$, $Y_0 \subset Y$ be closed subspaces. Assume that $T(X_0) \subset Y_0$. Denote by T^0 the restriction $T \mid X_0 \colon X_0 \to Y_0$ and by $\tilde{T}$ the operator $X/X_0 \to Y/Y_0$ induced by T. Then*

(i) *if $Y_0 \subseteq \operatorname{Im} T$, then $\tilde{T}$ is an embedding if and only if $T^{-1}(Y_0) = X_0$ and $\operatorname{Im} T$ is closed;*

(ii) *if $Y_0 \subset \operatorname{Im} T$ and $T^{-1}(Y_0) = X_0$, then T has closed range if and only if T is an embedding. Quantitatively*

$$\max(k(T^0), k(\tilde{T})) \leqslant k(T) \leqslant k(T^0) + \|T\|\, k(T^0)\, k(\tilde{T}) + k(\tilde{T}). \tag{1.1}$$

(iii) *if T^0 is onto and $\tilde{T}$ is an embedding, and if $R \subseteq L(X, Y)$ satisfies $R(X_0) \subset Y_0$, $\|R\| \leqslant (1-\delta)\gamma(T)$, $\delta > 0$, then $T + R$ has closed range, $(T+R)(X_0) = Y_0$, $(T+R)^{-1}(Y_0) = X_0$ and*

$$k(T+R) \leqslant 2\delta^{-1} k(T)(1 + \delta^{-1} k(T)\, \|T\|). \tag{1.2}$$

Proof. (i) is obvious, as well as the qualitative part of (ii). The inequality $k(T^0) \leqslant k(T)$ is clear since $\ker T^0 = \ker T$ (for $T^{-1}(Y_0) = X_0$). We apply Remark 1.1 to check inequality $k(\tilde{T}) \leqslant k(T)$. Let $[y] = y + Y_0 \in \operatorname{Im} T$ be such that $\operatorname{dist}(y, Y_0) < 1$. Choose $y_0 \in Y_0$ such that $\|y - y_0\| < 1$. Since $T(X_0) = Y_0$, $y - y_0 \in \operatorname{Im} T$ and there is $x_1 \in X$ *such that* $\|x_1\| < k(T)$ and $y - y_1 = Tx_1$. Also there is $x_0 \in X_0$ such that $Tx_0 = y_0$. Set $x = x_1 + x_0$. Then $y = Tx$, $[y] = \tilde{T}[x]$ and

$$\|[x]\| = \operatorname{dist}(x_1 + x_0, X_0) = \operatorname{dist}(x_1, X_0) \leqslant \|x_1\| \leqslant k(T).$$

Thus $k(\tilde{T}) \leqslant k(T)$, as required.

To prove the right-hand side inequality in (1.1), take $y \in \operatorname{Im} T$ such that $\|y\| < 1$; we will show that if $x \in X$ satisfies $Tx = y$ then

$$\operatorname{dist}(x, \ker T) \leqslant k(T^0) + \ker(\tilde{T}) + k(T^0)\, k(T)\, \|T\|. \tag{1.3}$$

Since $\operatorname{dist}(y, Y_0) < 1$ and $\ker \tilde{T} = [0] = [X_0]$ we have $\operatorname{dist}(x, X_0) < k(\tilde{T})$. Choose $x_0 \in X_0$ such that

$$\|x - x_0\| < k(\tilde{T}). \tag{1.4}$$

Then $\|y - Tx_0\| \leqslant \|T\|\, \|x - x_0\| \leqslant \|T\|\, k(\tilde{T})$ and so

$$\|Tx_0\| \leqslant \|y\| + \|T\|\, k(\tilde{T}) \leqslant 1 + \|T\|\, k(\tilde{T}). \tag{1.5}$$

Since $Tx_0 = T^0 x_0$, and by (1.5), there is $x' \in X_0$ such that $Tx' = Tx_0$ and

$$\|x'\| \leqslant k(T^0)(1 + \|T\|\, k(\tilde{T})).$$

By this and (1.4) $\|x-(x_0-x')\| \leqslant \|x-x_0\| + \|x'\| \leqslant k(\tilde{T})+k(T^0)+\|T\|\, k(T^0)\, k(\tilde{T})$. Since $x_0-x' \in \ker T$, the estimate (1.3) holds, and so does (1.1).

(iii) Let $R^0=R \mid X_0 : X_0 \to Y_0$ and $\tilde{R}(x+X_0)=y+Y_0$, $\tilde{R} \in L(X/X_0, Y/Y_0)$. By part (ii) $\gamma(\tilde{T}) \geqslant \gamma(T)$ and $\gamma(T^0) \geqslant \gamma(T)$. By the well-known property of epimorphisms and embeddings $\gamma(T^0+R^0) \geqslant \gamma(T^0)-\|R^0\| \geqslant \gamma(T)-\|R\|$ and $\gamma(\tilde{T}+\tilde{R}) \geqslant \gamma(\tilde{T})-\|\tilde{R}\| \geqslant \gamma(T)-\|R\|$. Thus if $\|R\| \leqslant (1-\delta)\,\gamma(T)$, then $(T+R)^0$ is an epimorphism, $(T+R)^{\sim}$ is an embedding, and both $k(T+R)^0)$ and $k((T+R)^{\sim})$ do not exceed $\delta^{-1}k(T)$. By (ii) $T+R$ has closed range and by (1.1) the inequality (1.2) holds. Q.E.D.

The next lemma generalizes Taylor [8], Lemma 2.1.

LEMMA 1.9. *Let X, Y, Z be Banach spaces. Denote*

$$K(X, Y, Z)=\{(T, S) \in L(X, Y) \times L(Y, Z) \colon S \cdot T=0\};$$

$$U(X, Y, Z)=\{(T, S) \in K(X, Y, Z) \colon \operatorname{Im} T=\ker S \textit{ and} \operatorname{Im} S \textit{ is closed}\}.$$

Then $U(X, Y, Z)$ is open in $K(X, Y, Z)$ and functions $k(T)$ and $k(S)$ are continuous on $U(X, Y, Z)$.

Proof. Let $(T_0, S_0) \in U(X, Y, Z)$. Let $\varepsilon>0$ satisfy the inequality

$$k(T_0)+k(S_0)+\varepsilon k(T_0)\, k(S_0)<1/\varepsilon. \tag{1.6}$$

(1) First, we show that if (1.6) and $\|T-T_0\|$, $\|S-S_0\|<\varepsilon$, $(T, S) \in M(X, Y, Z)$, then $\operatorname{Im} T=\ker S$. We will apply Lemma 1.2. If $y \in \ker S$, $\|y\|<1$, then $\|S_0 y\|=\|(S_0-S)\, y\|<\varepsilon$. So there is $y' \in Y$ such that $S_0 y'=S_0 y$ and $\|y'\|<k(S_0)\varepsilon$. Since $y-y' \in \ker S_0$ and $\|y-y'\|<1+\varepsilon k(S_0)$, there is $x_0 \in X$ such that $\|x_0\|<k(T_0)(1+\varepsilon k(S_0))$ and $Tx_0=y-y'$. Since

$$\begin{gathered} Tx_0-y=(T_0x_0-(y-y'))+(T-T_0)\, x_0-y'=(T-T_0)\, x_0-y', \\ \|Tx_0-y\| \leqslant \varepsilon\, \|x_0\|+\|y'\| \leqslant \delta:=\varepsilon(k(T_0)+k(S_0)+k(T_0)\, k(S_0)). \end{gathered} \tag{1.7}$$

If (1.6), then $\delta<1$ and by Lemma 1.2 $\operatorname{Im} T$ is a closed subspace of $\ker S$ and so, by the above proof, equal to $\ker S$.

(2) We will show now that if (1.6) and if $\|T-T_0\|<\varepsilon$, $\|S-S_0\|<\varepsilon$, then $\operatorname{Im} S$ is closed as well. By Proposition 1.7 $\operatorname{Im} S_0^*=\ker T_0^*$ and $\operatorname{Im} T_0^*$ is closed. Since (1.6) holds also for conjugated operators (by Remark 1.6), we can apply part (1) in this new setting and obtain that $\operatorname{Im} S^*=\ker T^*$. In particular, S^* has closed range and so does S.

Observe further that if δ is defined by (1.7) and ε satisfies (1.6), then it follows from the estimates in (1) and Lemma 1.2 that $k(T) \leqslant$

$k(T_0)(1+\varepsilon k(S_0))/(1-\delta)$, if $\|T-T_0\|$, $\|S-S_0\|<\varepsilon$, and so function $k(T)$ is locally bounded on $U(X, Y, Z)$. Applying the same argument as in (2) to conjugated operators, we get that $k(S^*)$ is locally bounded on $U(X, Y, Z)$. Thus the functions $k(T)$ and $k(S)$ are continuous on $U(X, Y, Z)$ (by Theorem 1.4). Q.E.D.

LEMMA 1.10. *Let E and Y be Banach spaces and $C \in L(E, Y^*)$. Assume that* $\operatorname{Im} C$ *is a w^*-closed subspace of Y^*. Then the operator $B = C^* \mid Y\colon Y \to E^*$ (where Y is considered as canonically embedded in Y^{**}) has closed range.*

Proof. Let Z denote the quotient space $Y/\operatorname{An}(\operatorname{Im} C)$, where $\operatorname{An}(\operatorname{Im} C)$ denotes the annihilator of $\operatorname{Im} C$ in Y and let π denote the canonical surjection of Y onto Z. Since $\operatorname{Im} C$ is w^*-closed, it is equal to the annihilator of $\operatorname{An}(\operatorname{Im} C)$ in Y^*, and the latter is equal to $\pi^*(Z^*)$. Thus $\operatorname{Im} C = \pi^*(Z^*)$. Because of this we can represent C as $\pi^* \circ C_1$, where C_1 is an epimorphism in $L(E, Z^*)$. Then $C^* = C_1^* \circ \pi^{**}$, where $C_1^* \in L(Z^{**}, E^*)$ is an embedding. If we set $B_1 = C_1 \mid Z\colon Z \to E^*$, then B_1 is also an embedding and $B = B_1 \circ \pi$. Since π is a surjection, B has closed range. Q.E.D.

2. LINEAR PENCILS OF OPERATORS WITH CLOSED RANGES

The proof of Theorem 1 (to be given in Sect. 3), relies on the next result.

THEOREM 2.1. *Let $G \subset \mathbb{C}$ be a connected open set and $S, V \in L(X, Y)$. Denote $T_z = S - zV$, $z \in G$ and assume that there is a Banach space F and a function $B \in H(G, L(Y, F))$ such that* $\operatorname{Im} T_z = \ker B(z)$ *for every $z \in G$. Then there are closed subspaces $X_0 \subset X$ and $Y_0 \subset Y$ such that for all $z \in G$,*

$$T_z(X_0) = Y_0 \qquad \textit{and} \qquad T_z^{-1}(Y_0) = X_0.$$

Consequently, for every $z \in G$ the operator $T_z \mid X_0 \in L(X_0, Y_0)$ is an epimorphism, while the operator $\tilde{T}_z\colon X/X_0 \to Y/Y_0$, induced by T_z, is an embedding.

The construction of subspaces X_0 and Y_0 utilizes the FSC which we define more generally than needed in this section.

DEFINITION 2.2. Let X and Y be complex linear spaces and $T(z)$ be a formal power series in $z_1, z_2, \ldots, z_n$, whose coefficients are linear transformations from X to Y. We represent it as formal series $T(z) = \sum_{k=0}^{\infty} T_k(z)$, where $T_k(\cdot)$ are homogeneous polynomials. We say that $T(z)$ satisfies FSC if

for every $k \geqslant 1$ and for every X-valued polynomial $x(z)$ of degree smaller than k, if $y(z) = T(z)\,x(z)$ is a formal power series in $z_1, z_2, \ldots, z_n$ whose all terms of degree smaller than k vanish, then all the coefficients of $y(z)$ corresponding to terms of degree k belong to $\operatorname{Im} T(0) = \operatorname{Im} T_0$.

If $T \in H(G, L(X, Y))$, $G \subset \mathbb{C}^n$, we say that the function $T(z)$ satisfies FSC at $z = a$ if the power series in $z_1 - a_1, \ldots, z_n - a_n$ representing $T(z)$ satisfies FSC (in the sense of formal power series).

Of couse, if the costant term $T(0) = T_0$ is an epimorphism then the power series $T(z)$ satisfies FSC. It is less obvious but also easy to check that $T(z)$ satisfies FSC if $T(0)$ is one-to-one (for equation $T(z)\,x(z) \equiv 0$ implies $x(z) \equiv 0$, if $x(z)$ is a formal power series).

The FSC may seem complicated, but it is just a precise formulation that the power series equation $T(z)\,x(z) = 0$ can be solved inductively for $x(z)$, with finite number of derivatives at $z = 0$ prescribed. If, in addition, $T(z)$ is analytic and $\operatorname{Im} T(0)$ closed then the formal solution $x(z)$ can be constructed in a controlled way so that it is holomorphic near $z = 0$ (see Proof of Lemma 4.1(iii) $\Rightarrow$ (iv)). The technique of the next lemma is much the same as that of Taylor [8, Lemma 2.3], however, the realization of the role of FSC in this context seems to be new.

LEMMA 2.3. *Let X, Y, and F be complex linear spaces and let $T(z)$ and $B(z)$ be formal power series in $z_1, \ldots, z_n$ whose coefficients are linear transformations from X to Y and Y to Z, respectively. Assume that* $\operatorname{Im} T(0) = \ker B(0)$ *and that* $B(z) \circ T(z) \equiv 0$. *Then $T(z)$ satisfies* FSC.

Proof. Let $x(x) = x_0 + x_1(z) + x_2(z) + \cdots + x_{k-1}(z)$ be an X-valued polynomial of degree $\leqslant k - 1$ (written as sum of homogeneous polynomials) such that $T(z)\,x(z) = y_k(z) +$ terms of degree greater than k, and $y_k(z)$ is a homogeneous polynomial of degree k. We have show that coefficients of $y_k(z)$ belong to $\operatorname{Im} T_0$, or, equivalently, that $B(0)\,y_k(z) = 0$ for every $z \in \mathbb{C}^n$.

If we represent $B(z) = \sum_{i=0}^{\infty} B_i(z)$, where $B_i(z)$ is a homogeneous polynomial of degree i, then we can sum up our assumptions as follows:

$$y_k(z) = \sum_{i=1}^{k} T_i(z)\,x_{k-i}(z);$$

$$\sum_{i+j=l} T_i(z)\,x_j(z) = 0 \qquad \text{for} \quad 0 \leqslant 1 < k,\ i, j \geqslant 0; \tag{2.1}$$

$$B_0 T_i(z) = -\sum_{s=1}^{i} B_s(z)\,T_{i-s}(z), \qquad i \geqslant 0.$$

Substituting the last formula to the equation $B_0 y_k(z) = \sum_{i=1}^{k} (B_0 T_i(z)) x_{k-i}(z)$ we get

$$\begin{aligned} B_0 y_k(z) &= -\sum_{i=1}^{k} \sum_{s=1}^{i} B_s(z) T_{i-s}(z) x_{k-i}(z) \\ &= -\sum_{s=1}^{k} B_s(z) \left(\sum_{i=s}^{k} T_{i-s}(z) x_{k-i}(z) \right) = 0. \end{aligned}$$

(by (2.1)). Q.E.D.

LEMMA 2.4. *Let $S, V \in L(X, Y)$. Define by induction subspaces X'_n, Y'_n, $n \geqslant 0$, so that $X'_0 = \ker S$, $Y'_n = V(X'_n)$, $X'_{n+1} = S^{-1}(Y'_n)$ and set*

$$X_0 = \mathrm{Cl}\left(\bigcup_{n=0}^{\infty} X'_n \right); \qquad Y_0 = \mathrm{Cl}\left(\bigcup_{n=0}^{\infty} Y'_n \right).$$

Assume that the pencil $z \to S - zV$ satisfies FSC *at $z = 0$ and that S has closed range. Then*

$$V(X_0) \subset Y_0, \qquad Y_0 \subset \mathrm{Im}\, S, \tag{2.2}$$

$$S^{-1}(Y_0) = X_0, \qquad S(X_0) = Y_0. \tag{2.3}$$

Proof. We prove first by induction on n that

$$X'_n \subset X'_{n+1} \qquad \text{and} \qquad Y'_n \subset Y'_{n+1}, \quad n \geqslant 0. \tag{2.4}$$

Since $S^{-1}(0) \subset S^{-1}(V(S^{-1}(0)))$ and $V(S^{-1}(0)) \subset VS^{-1}(V(S^{-1}(0)))$, inclusions (2.4) hold for $n = 0$. If we assume they are fullfilled for n, then they also hold for $n+1$, since $X'_{n+1} = S^{-1}(Y'_n) \subset S^{-1}(Y'_{n+1}) = X'_{n+2}$, and so $Y'_{n+1} = V(X'_{n+1}) \subset V(X'_{n+2}) = Y'_{n+2}$, as required.

By (2.4), $X'_\infty = \bigcup_{n=0}^{\infty} X'_n$ and $Y'_\infty = \bigcup_n Y'_n$ are linear manifolds. Of course $X_0 = \mathrm{Cl}(X'_\infty)$, $Y_0 = \mathrm{Cl}(Y'_\infty)$, and since $V(X'_n) \subset Y'_n$, we have $V(X_0) \subset Y_0$.

The FSC condition is needed to show that $Y'_n \subset \mathrm{Im}\, S$, $n \geqslant 0$ (and is actually equivalent to the latter fact). Namely, a closer inspection of the definitions of X'_n and Y'_n reveals that $y_n \in Y'_n$ if and only if there exist vectors $x_0, ..., x_n \in X$ such that $Sx_0 = 0$, $Sx_1 = Vx_0, ..., Sx_n = Vx_{n-1}$, and $y_n = Vx_n$. These relations imply that $(S - zV)(x_0 + x_1 z + \cdots + x_n z^n) = y_n z^n +$ terms of degree greater than n. Thus by FSC $y_n \in \mathrm{Im}\, S$ and so $Y'_\infty \subset \mathrm{Im}\, S$. Since the latter subspace is closed,

$$Y_0 \subset \mathrm{Im}\, S. \tag{2.5}$$

Since $X'_{n+1} = S^{-1}(Y'_n)$ and by (2.4), $X'_\infty = S^{-1}(Y'_\infty)$. Then of course $\mathrm{Cl}(X') \subset S^{-1}(\mathrm{Cl}(Y'_\infty))$, but it takes closed range property of S to show the reverse inclusion. Let $M > k(S)$. If $x \in S^{-1}(Y_0)$ then $Sx \in \mathrm{Cl}(Y'_\infty)$ and there are vectors $y_n \in Y'_\infty$, $n \geqslant 0$, such that $\sum_n \|y_n\| < \infty$ and $\sum y_n = Sx$. By (2.5) $y_n \in \mathrm{Im}\, S$ and so there are x_n, $\|x_n\| \leqslant M \|y_n\|$ such that $Sx_n = y_n$. Clearly $x_n \in X'_\infty$, $\tilde{x} = \sum_n x_n$ belongs to X_0 and $S\tilde{x} = y$. Thus $S(x - \tilde{x}) = 0$, and since $\ker S \subset X_0$, $x \in X_0$. We have thus proved that $S^{-1}(Y_0) = X_0$, which, together with (2.5) yields $S(X_0) = Y_0$. Q.E.D.

COROLLARY 2.5. *Let $z \to T_z$: $\mathbb{C} \to L(X, Y)$ be a linear pencil satisfying FSC at $z = a$. Assume that T_a has closed range. Then there are closed subspaces $X_0 \subseteq X$ and $Y_0 \subseteq Y$, and $r > 0$ such that $T_z(X_0) \subseteq Y_0$ for all $z \in \mathbb{C}$ and*

$$T_z(X_0) = Y_0 \qquad \text{for} \quad |z - a| \leqslant r, \tag{2.6}$$

$$Y_z^{-1}(Y_0) = X_0 \qquad \text{for} \quad |z - a| \leqslant r. \tag{2.7}$$

More specifically, for any $r > 0$, such that

$$\sup_{|z-a| \leqslant r} \|T_z - T_a\| \leqslant (1 - \delta)\, \gamma(T_a).$$

for some $\delta > 0$, Eqs. (2.6), (2.7) *hold, as well as*

$$\sup_{|z-a| \leqslant r} k(T_z) \leqslant 2\delta^{-1} k(T_a)(1 + \delta^{-1} k(T_a) \|T_a\|) < \infty. \tag{2.8}$$

Proof. We can represent $T_z = S - (z - a)V$, with $S = T_a$ and apply Lemma 2.4 to get subspaces $X_0 \subset X$ and $Y_0 \subset Y$ satisfying (2.2) and (2.3). Then one can apply Lemma 1.8(iii) to get (2.6)-(2.8). We omit further details. Q.E.D.

Proof of Theorem 2.1. Choose $a \in G$. By Lemma 2.3 the pencil T_z satisfies FSC at any point, e.g., at $z = a$, and so by Corollary 2.5 there is a disc $D = D(a, r)$, $r > 0$ such that

$$T_z(X_0) = Y_0, \qquad T_z^{-1}(Y_0) = X_0 \quad \text{for } z \in D.$$

Since $Y_0 \subset \mathrm{Im}\, T_z = \ker B(z)$ for $z \in D$, the analytic function $z \to B(z) \mid Y_0 \colon G \to L(Y_0, F)$ vanishes on D and so on G; thus

$$Y_0 \subseteq \mathrm{Im}\, T_z \qquad \text{for} \quad z \in G. \tag{2.9}$$

By this and Lemma 1.8(i), $T_z^{-1}(Y_0) = X_0$ if and only if $\tilde{T}_z$ is an embedding, where $\tilde{T}_z \colon X/X_0 \to Y/Y_0$ is the operator induced by T_z. Therefore the set

$$\tilde{G} = \{z \in G \colon T_z^{-1}(T_0) = X_0\} = \{z \in G \colon \tilde{T}_z \text{ is an embedding}\}$$

is open. We will show that it is closed in G. Let $b \in G \cap \mathrm{Cl}(\tilde{G})$. Since T_z satisfies FSC at $z = b$, by Corollary 2.5 there is a disc $D' = D(b, r')$ such that $\sup\{k(T_z): z \in \bar{D}'\} < \infty$. (This follows also from Lemma 1.9.) Choose a sequence $z(n) \in G \cap D'$ such that $\lim z(n) = b$. Since $\tilde{T}_{z(n)}$ are embeddings and $\sup k(\tilde{T}_{z(n)}) \leqslant \sup k(T_{z(n)}) < \infty$ (by (1.1)), therefore by Corollary 1.5 $\tilde{T}_b$ is an embedding and so $b \in \tilde{G}$. Seeing that $\tilde{G}$ is nonempty, closed and open in G, which is connected, $\tilde{G} = G$. Thus $T_z^{-1}(Y_0) = X_0$ for all $z \in G$ and by (2.9) $Y_0 = T_z(T_z^{-1}(Y_0)) = T_z(X_0)$ for all $z \in G$. Q.E.D.

3. Operators between Spaces of Analytic Functions with Absolutely Convergent Taylor Series

We are now ready to prove Theorem 1 using Theorem 2.1. The latter reduces the treatment of linear families of operators with closed ranges to families of embeddings and surjections considered separately, and in these special cases Theorem 1 is proven easily. Then the linearization technique and induction on dimension settles the general case.

Remark 3.1. Operator $\tilde{T}$ in Theorem 1 is a well-defined bounded operator from $A_+(P, X)$ to $A_+(P, Y)$. Namely, if $T \in H(G, L(X, Y))$, then $T \mid P \in A_+(P, L(X, Y))$ and $\tilde{T}f$, $f \in A_+(P, X)$, is a function in $A_+(P, Y)$, whose Taylor expansion in $z_1, \ldots, z_n$ is formal convolution of power series of $T \mid P$ and of f. Thus the norm of T is smaller than or equal to norm of $T \mid P$ in $A_+(P, L(X, Y))$ (in the sense of (2)). To avoid ambiquity we will frequently denote $\tilde{T}$ by $A_+(T)$.

LEMMA 3.2. *Theorem* 1 *holds if* $n = 1$ *and every* $T(z)$, $z \in P$, *is an epimorphism.*

Proof. Let E be an l^1 space of sufficiently large cardinality so that there is an epimorphism $\pi: E \to Y$. Since $T(z)$ are onto, by Leiterer [4, Lemma 3.1] there exists $A \in H(G, L(E, X))$ $(G \subset \mathbb{C}^1)$ such that $T(z) A(z) = \pi$ for all $z \in G$. We have to show that $\operatorname{Im} A_+(T) = A_+(P, Y)$. Since $A_+(T) \circ A_+(A) = A_+(\pi)$, it remains to observe that $A_+(\pi)$ is onto.

We can assume that $P = \bar{D}(0, R)$ $(n = 1)$. Let $M > k(\pi)$. If $g \in A_+(P, Y)$, then $g(z) = \sum_{k=0}^{\infty} y_k z^k$, where $\sum_{k=0}^{\infty} \|y_k\| R^k < \infty$. We can find $x_k \in E$ such that $\pi(x_k) = y_k$ and $\|x_k\| \leqslant M \|y_k\|$. Then the series $\sum_{k=0}^{\infty} x_k z^k$, $|z| \leqslant R$ defines $f \in A_+(P, E)$ such that $\pi(f(z)) = g(z)$. Thus $A_+(\pi)$ is onto. Q.E.D.

LEMMA 3.3. *Theorem* 1 *holds if all* $T(z)$, $z \in G$, *are embeddings and* $n = 1$.

Proof. In the special case under consideration all $T(z)^* \in L(Y^*, X^*)$, $z \in G$, are epimorphisms. Therefore, similarly as in the last lemma there exist: an l^1 space E, an epimorphism $\pi: E \to X^*$ and a function

$A_1 \in H(G, L(E, X^*))$ such that $T(z)^* \circ A_1(z) = \pi$ for all $z \in G$. Now, set $F = E^*$, $j = \pi^* \mid X \in L(X, F)$, and $C(z) = A_1(z)^* \mid Y$, $z \in G$, where X and Y are considered as embedded canonically in X^{**} and Y^{**}, respectively. Then j is an embedding, $C \in H(G, L(Y, F))$ and $B(z) \circ T(z) = j$, for all $z \in G$.

The inclusion ($\subset$) in Eq. (3) in Theorem 1 is obvious. Conversely, let $g \in A_+(P, Y)$ and $g(z) \in \operatorname{Im} T(z)$ for every $z \in P$. Set $f(z) = C(z)\, g(z)$, $z \in P$. By Remark 3.1 function f belongs to $A_+(P, E)$. Since $g(z) \in \operatorname{Im} T(z)$, $z \in P$, we get $f(z) \in \operatorname{Im} C(z) \circ T(z) = \operatorname{Im} j$, $z \in P$. The operator j being an embedding, there is a unique function $h\colon P \to X$ such that $j(h(z)) = f(z)$; of course $h \in A_+(P, X)$. Eventually, $C(z)(T(z)\, h(z) - g(z)) = j(h(z)) - f(z) = 0$, and since $\ker C(z) \cap \operatorname{Im} T(z) = (0)$ for each $z \in G$, we infer that $T(z)\, h(z) - g(z) \equiv 0$ in P. Thus $g \in \operatorname{Im} A_+(T)$. Q.E.D.

Proof of Theorem 1. (1) We start from the case $n = 1$; afterwords we will continue by induction on n.

Assertion 1. Theorem 1 is true if $n = 1$ and $T(z) = S - zV$ (in addition to the other assumptions).

Without loss of generality G can be replaced by its connected open component containing P. This and assumption about $B(z)$ allow us to apply Theorem 2.1. Thus there exist subspaces $X_0 \subset X$ and $Y_0 \subset Y$ such that all operators $T^0(z) = T(z) \mid X_0$, $z \in G$, are epimorphisms onto Y_0, and all operators $\tilde{T}(z) \in L(\tilde{X}, \tilde{Y})$, $z \in G$, induced by $T(z)$, are embeddings ($\tilde{X} = X/X_0$, $\tilde{Y} = Y/Y_0$).

The inclusion ($\subset$) in Eq. (3) is trivial. To prove the reverse inclusion, take arbitrary $g \in A_+(P, Y)$ such that $g(z) \in \operatorname{Im} T(z)$, $z \in G$. Let $\pi\colon X \to \tilde{X}$ and $\sigma\colon Y \to \tilde{Y}$ be canonical surjections, and $i\colon X_0 \to X$, $j\colon Y_0 \to Y$, be canonical embeddings. Then $\sigma g \in A_+(P, \tilde{Y})$ and $\sigma g(z) \in \operatorname{Im} \tilde{T}(z)$ for every $z \in G$. Since all $\tilde{T}(z)$ are embeddings, by Lemma 3.3 there is $h \in A_+(P, \tilde{X})$ such that $\tilde{T}(z)\, h(z) = \sigma g(z)$, $z \in P$. Since π is an epimorphism, by Lemma 3.2, $A_+(\pi)$ maps $A_+(P, X)$ onto $A_+(P, \tilde{X})$, and so there is $f_1 \in A_+(P, X)$ such that $\sigma g(z) = \tilde{T}(z)\, \pi f(z)$ for every $z \in P$. Since $\sigma T(z) = \tilde{T}(z)\pi$, for $z \in G$, we get $\sigma(g(z) - T(z) f_1(z)) = 0$ and so $h_1(z) := g(z) - T(z) f_1(z) \in \operatorname{Im} j = Y_0$, $z \in P$, and $h_1 \in A_+(P, Y_0)$. All $T^0(z)$ being epimorphisms, Lemma 3.2 implies that there is a function $k \in A_+(P, X_0)$ such that $h_1(z) = g(z) - T(z)\, f_1(z) = T^0(z)\, k(z) = T(z)\, k(z)$, for $z \in P$. Therefore $g(z) = T(z)(f_1(z) + k(z))$, $z \in P$, where $k + f_1 \in A_+(P, X)$. The assertion is proved.

We will now show that general one dimensional case follows from the linear case. Recall after [1, 2] that operator valued functions $T, T_1 \in H(G, L(X, Y))$ are called analytically equivalent if there exist functions $F \in H(G, L(X))$ and $E \in H(G, L(Y))$, whose all values are invertible, such that $T_1(z) = E(z) \circ T(z) \circ F(z)$, $z \in G$.

LINEARIZATION THEOREM. *Let $T \in H(G, L(X, Y))$. Then there exist a Banach space Z and operators $S, V \in L(X \oplus Z, Y \oplus Z)$ such that the operator valued functions $z \to T(z) \oplus T_Z$ and $z \to S - zV$, $z \in G$, are analytically equivalent.* (See den Boer [1] and Gohberg, Kaashoek, and Lay [2] for details and background.)

Assume that $T \in H(G, L(X, Y))$ is as in Theorem 1, in particular there exist $B \in H(G, L(Y, F))$ such that $\operatorname{Im} T(z) = \ker B(z)$, $z \in G$. Set $B_1(z) := B(z) \oplus 0_Z \in L(Y \oplus Z, F \oplus Z)$; of course B_1 is an analytic function such that $\ker B_1(z) = \operatorname{Im}(T(z) \oplus I_Z)$ for $z \in G$. By the linearization theorem $T(z) \oplus I_Z = E(z) \circ (S - zV) \circ F(z)$, where E, F are as above. If we set $B_2(z) = B_1(z) \circ E(z)$, then it is clear that $B_2(\cdot)$ is an analytic operator-valued function in G such that $\ker B_2(z) = \operatorname{Im}(S - zV)$ for every $z \in G$. Therefore the linear pencil $S - zV$ satisfies all assumptions of Assertion 1 and so Eq. (3) holds, i.e.,

$$\operatorname{Im} A_+(S - (\cdot)V) = \{g \in A_+(P, Y \oplus Z) : g(z) \in \operatorname{Im}(S - zV) \text{ for } z \in P\}.$$

By the Assertion 2, which is formulated next, Eq. (3) holds also for the equivalent operator function $T(\cdot) \oplus I_Z$, i.e.,

$$\operatorname{Im} A_+(T \oplus I_Z) = \{g \in A_+(P, Y \oplus Z) : g(z) \in (\operatorname{Im} T(z)) \oplus Z, z \in P\}. \qquad (3.1)$$

Assertion 2. Let P be a compact polydisc in G and $T, T_1 \in H(G, L(X, Y))$ be analytically equivalent. If Eq. (3) holds for the function $T_1(\cdot)$, then it holds for $T(\cdot)$ as well.

(We omit the simple proof which follows from Proposition 3.1 and inspection of the ranges $\operatorname{Im} A_+(T) = \operatorname{Im}(A_+(E)) \circ A_+(T_1) \circ A_+(F))$.

We end part (1) of the proof with the observation that left-hand side in (3.1) is equal to $\operatorname{Im} A_+(T) \oplus A_+(P, Z)$, while the right-hand side is equal to

$$\{g \in A_+(P, Y) : g(z) \in \operatorname{Im} T(z), z \in P\} \oplus A_+(P, Z)$$

and so Eq. (3) follows.

(2) Having proved Theorem 1 for $n = 1$, we will show now that it holds for n, provided it is already known for all $k \leqslant n$.

We establish some notation. Points of $\mathbb{C}^n$ are represented as $z = (z', z_n)$, $z' \in \mathbb{C}^{n-1}$, and $P = Q \times \bar{D}$, where Q is a polydisc in $\mathbb{C}^{n-1}$ and $\bar{D}$ a closed disc. We let $G_n \subset \mathbb{C}$ be the maximal open set such that $Q \times G_n \subset G$; of course $\bar{D} \subset G_n$. Consider Banach spaces $\tilde{X} = A_+(Q, X)$, $\tilde{Y} = A_+(Q, Y)$, and $\tilde{F} = A_+(Q, F)$. The heart of our argument is the observation that the following natural (isometric isomorphic) identifications hold:

$$A_+(P, X) = A_+(D, \tilde{X}), \quad A_+(P, Y) = A_+(D, \tilde{Y}), \quad \text{and} \quad A_+(P, Z) = A_+(D, \tilde{Z}). \qquad (3.2)$$

Define furthermore, for each $z_n \in G_n$, operators $\tau(z_n) \in L(\tilde{X}, \tilde{Y})$ and $\beta(z_n) \in L(\tilde{Y}, \tilde{F})$ as the multiplication operators induced by the functions of $(n-1)$ arguments: $z' \to T(z', z_n)$ and $z' \to B(z', z_n)$, respectively (cf. Proposition 3.1). It is clear that $\tau(\cdot)$ and $\beta(\cdot)$ are analytic operator-valued functions in G_n.

Applying inductive assumptions to the function $z' \to T(z', z_n)$ (z_n-fixed) of $(n-1)$ arguments we get, by formula (3), that $\operatorname{Im} \tau(z_n) = \ker \beta(z_n)$, for each $z_n \in G_n$. Therefore the function $\tau \in H(G_n, L(\tilde{X}, \tilde{Y}))$ satisfies all the assumptions of Theorem 1, and so, by its one-dimensional version, the operator

$$A_+(\tau)\colon \quad A_+(D, \tilde{X}) \to A_+(D, \tilde{Y})$$

has closed range. Furthermore

$$\operatorname{Im} A_+(\tau) = \{\gamma \in A_+(D, \tilde{Y}) \colon \gamma(z_n) \in \operatorname{Im} \tau(z_n), z_n \in D\}. \tag{3.3}$$

Now, the identifications (3.2) identify also operator $A_+(T)$ with $A_+(\tau)$ (and $A_+(B)$ with $A_+(\beta)$). With the use of these identifications Eq. (3) is just a translation of (3.3). Q.E.D.

Remark 3.4. Equation (3) in Theorem 1 is equivalent to the relation

$$\operatorname{Im} A_+(T) = \ker A_+(B)$$

(for $\operatorname{Im} T(z) = \ker B(z)$ for all $z \in P$).

The next corollary (together with results of Sect. 5) is the main application of this paper to the study of Taylor spectrum which is carried over in [8]. The corollary follows immediately from the last remark and Theorem 1.

COROLLARY 3.5. *Let $P \subset \mathbb{C}^k$ be a compact polydisc, G its open neighbourhood $X^0, X^1, \dots, X^m$ be complex Banach spaces and $d^i \in H(G, L(X^i, X^{i+1}))$, $0 \leqslant i \leqslant m-1$. Assume that for each $z \in G$ the complex*

$$0 \to X^0 \to X^1 \to \cdots \to X^m \to 0, \tag{3.4}$$

where the differentials are $0, d^0(z), d^1(z) \cdots d^{m-1}(z), 0$; respectively, is exact. Then the complex

$$0 \to A_+(P, X^0) \to A_+(P, X^1) \to \cdots \to A_+(P, X^{m-1}) \to A_+(P, X^m) \to 0,$$

whose differentials are $0, A_+(d^0), A_+(d^1), \dots, A_+(d^{m-1}), 0$ is exact as well.

4. Characterization of Analytic Families of Operators with Regularly Varying Ranges

The proof of the characterizations formulated in Theorem 2 depends on the next lemma (which is a local version of the theorem) and results of Leiterer [4] on Banach coherent analytic Frechet sheaves (BCAF sheaves).

LEMMA 4.1. *Let $a \in U \subset \mathbb{C}^k$, U be open and $T \in H(U, L(X, Y))$. Then the following conditions are equivalent*

(i) $\limsup_{z \to a} k(T(z)) < \infty$;

(ii) *when* $z_n \to a$, $x_n \in \operatorname{Im} T(z_n)$ *and* $\|x_n - x_0\| \to 0$, *then* $x_0 \in \operatorname{Im} T(a)$;

(iii) FSC *holds at* $z = a$ *and* $\operatorname{Im} T_a$ *is closed*;

(iv) *there is a Banach space E, a neighbourhood V of a in U and a function $A \in H(V, L(E, X))$ such that* $\operatorname{Im} A(z) = \ker T(z)$ *for* $z \in V$;

(v) *there is a Banach space F, a neighbourhood V of a in U and a function $B \in H(V, L(Y, F))$ such that* $\ker B(z) = \operatorname{Im} T(z)$ *and* $\operatorname{Im} B(z)$ *is closed for all* $z \in V$.

Since both (i) and (ii) hold in some neighbourhood, condition (iii) has the same property. This is rather surprising since neither FSC nor the property of having closed range, taken separately, have to hold in any neighbourhood.

COROLLARY 4.2. *Let T be an analytic operator-valued function in $U \subset \mathbb{C}^n$. Then the set of all $a \in U$ such that* $\operatorname{Im} T(a)$ *is closed and function $T(z)$ has* FSC *at $z = a$ is open in U.*

Proof of Lemma 4.1. We first prove the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i); then we will show that also condition (v) is equivalent to (i). We may assume without loss of generality that $a = 0$.

(i) $\Rightarrow$ (ii) by Theorem 1.4(iii).

(ii) $\Rightarrow$ (iii) Let $x(z)$ be an X-valued polynomial of degree smaller than n ($n \geq 1$) such that $T(z)\,x(z)$ is a formal power series in $z_1, \ldots, z_k$ (with coefficients in X) without terms of degree lower than n. Let $y(z)$ denote the homogeneous part of $T(z)\,x(z)$ of degree n. It is enough to show that $y(z) \in \operatorname{Im} T(0)$ for every $z \in \mathbb{C}^k$.

Fix $u \in \mathbb{C}^k$, $u \neq 0$ and consider $f(\lambda) = \lambda^{-n}(T(\lambda u)\,x(\lambda u))$, $\lambda \neq 0$. Then $f(\lambda)$ is a holomorphic function in a pointed neighbourhood of 0 and clearly $\lim_{\lambda \to 0} f(\lambda) = y(u)$. Since $f(\lambda) \in \operatorname{Im} T(\lambda u)$, $\lambda \neq 0$, condition (ii) implies that $y(u) \in \operatorname{Im} T(0)$, as required.

(iii) $\Rightarrow$ (iv) If $x(\cdot)$ (or $T_n(\cdot)$) is a homogeneous polynomial with vector (or operator) coefficients, then we denote by $|x|$ (or $|T_n|$) the sum of the norms of all coefficients. Let $\bar{D}(0, R_i)^k$ be a closed polydisc contained in U and let $T(z)=\sum_{n=0}^{\infty} T_n(z)$, $(T_0(z)=T_0)$, represents $T(z)$ by homogeneous polynomials. Cauchy formula and combinatorial argument show that if $\|T(z)\| \leqslant C$ for $\max(|z_i|) \leqslant R_1$ then $|T_n| \leqslant CR^{-n}$, if R is chosen so that $R < R_1/k$. Denote $P=\bar{D}(0, R)^k$.

Assertion. Under the above denotations and assumptions, if $k(T_0) < M$ then for every $x \in \ker T_0$ there is a power series $x(z)=\sum_{n=0}^{\infty} x_n(z)$ (where $x_n(z)$ are homogeneous polynomials) such that: (i) $x_0=x(0)=x$, (ii) $T(z)\,x(z)=0$ (as formal power series), and (iii) $|x_n| \leqslant CM(1+CM)^{n-1}R^{-n}\|x\|$, $n \geqslant 1$.

The assertion generalizes Taylor [8, Lemma 2.2]; the specific estimate (iii) extends that of [7, Lemma 1.6]. We construct homogeneous polynomials $x_n(z)$ by induction on n, starting from $x_0=x$. Assume that $x_0, x_1(z),..., x_{n-1}(z)$ are already constructed so that (iii) holds and $\sum_{i=0}^{s} T_i(z)\,x_{s-i}(z)=0$ for $0 \leqslant s \leqslant n-1$. We will construct a homogeneous polynomial $x_n(z)$ satisfying the same conditions for n. The inductive assumption means that the series $T(x)\,x(z)$, where $x(z)=\sum_{i=0}^{n-1} x_n(z)$, does not have terms of degree smaller than n. Since $T(z)$ satisfies FSC at $z=0$, the homogeneous polynomial $y(z)=\sum_{i=1}^{n} T_i(z)\,x_{n-i}(z)$, of degree n, has all coefficients in Im T_0. Since $k(T_0) < M$, there is a homogeneous, nth degree polynomial $x_n(z)$, with coefficients in X, such that $T_0 x_n(z) = -y(z)$ and $|x_n| \leqslant M\,|y|$. The first condition reads

$$\sum_{i=0}^{n} T_i(z)\,x_{n-i}(z)=0. \tag{4.1}$$

Using the inequalities $|y| \leqslant \sum_{i=1}^{n} |T_i|\,|x_{n-i}|$, (iii) for $0, 1,..., n-1$ and $|T_i| \leqslant CR^{-i}$, we obtain eventually

$$|x_n| \leqslant \frac{MC}{1+MC}((1+MC)/R)^n,$$

and so (iii) holds for n. The induction is completed.

We use the assertion to obtain (iv). Let E be an l^1-space of large enough cardinality so that there is $A_0 \in L(E, X)$ such that Im $A_0=\ker T_0$. We can also assume $\|A_0\|=1$. If $\{e_t\}$ is the canonical basis of $l^1=E$, we set $x^t=A_0 e_t$, $\|x_t\|=1$. Denote in the Assertion $C_1=MC/(1+MC)$ and $r=R/(1+MC)$. Then for every t there exist a sequence of homogeneous polynomials $x_n^t(z)$, $n=0, 1,...$, such that $x_0^t=x_t$, $|x_n^t| \leqslant C_1 r^{-n}$ and

$$\sum_{i=0}^{n} T_i(z)\,x_{n-i}^t(z)=0, \qquad n=0, 1, 2,..., t \text{ arbitrary}. \tag{4.2}$$

If $e=\sum \alpha_i e_i \in E$, we set $A_n(z)=\sum \alpha_i x'_{n-i}(z)$. Then A_n (z) is a homogeneous polynomial with coefficients in $L(E, X)$ and

$$|A_n| \leqslant C_1 r^{-n}. \tag{4.3}$$

Set $A(z)=\sum_{n=0}^{\infty} A_n(z)$, as a formal power series. By (4.2), $A(z)$ is an analytic function in $D(0, r)^k$, and by (4.2), $T(z) \circ A(z)=0$ in this polydisc. Of course $A(0)=A_0$ and, by construction, $\operatorname{Im} A_0=\ker T_0$. These properties and Lemma 1.9 imply that there is a neighbourhood V of $z=0$ such that $\operatorname{Im} A(z)=\ker T(z)$ for $z \in V$ and $\sup_{z \in V} k(A(z))< +\infty$, and so (iv) holds.

(iv) $\Rightarrow$ (i) by direct application of Lemma 1.9.

Having thus shown the equivalence of conditions (i)–(iv), we can apply it now to the function $T(z)^*$. Of course condition (i) holds for $T(\cdot)$ if and only if it does for the function $z \rightarrow T(z)^*$. Therefore the function $T(\cdot)^*$ satisfies also condition (iv), and so there exists a neighbourhood V of $z=0$, an l^1 space E and a function $C \in H(V, L(E, Y^*))$ such that $\operatorname{Im} C(z)= \ker T(z)^*$ and $\operatorname{Im} T(z)$ is closed for $z \in V$. Define $B(z)=C(z)^* \mid Y$, Y being canonically embedded in Y^{**}. Of course $B \in H(V, L(Y, E^*))$. Since $\operatorname{Im} C(z)=\ker T(z)^*$, $\operatorname{Im} C(z)$ is w^*-closed in Y^*, and so Lemma 1.10 implies that $B(z)$ has closed range for $z \in V$. To complete the proof of (i) $\Rightarrow$ (v) we have to show yet that $\operatorname{Im} T(z)=\ker B(z)$. Since $\operatorname{Im} C(z)= \ker T(z)^*$ and $\operatorname{Im} T(z)^*$ is closed for $z \in V$, Proposition 1.7 implies that $\operatorname{Im} T(z)^{**}=\ker C(z)^*$. We omit the argument showing that $\operatorname{Im} T(z)^{**} \cap Y=\operatorname{Im} T(z)$. Since $B(z)=C(z)^* \mid Y$, we can conclude that $\operatorname{Im} T(z)=\ker B(z)$.

The converse implication (v) $\Rightarrow$ (i) is a direct consequence of Lemma 1.9.

Proof of Theorem 2. Since conditions (i)–(iii) are of local character, their equivalence follows from Lemma 4.1, and by this lemma, (iv) implies any of the first three as well.

(i) $\Rightarrow$ (iv) By Lemma 4.1 condition (iv) holds locally. The global version will be obtained by applying Leiterer's theory of BCAF sheaves; cf. [4]. To save space, however, we have to refer the reader to the cited paper for precise formulations of notions and results we use.

Define the kernel sheaf, KT, to be the analytic Frechet sheaf; cf. Leiterer [4, Sect. 1], with

$$KT(U)=\{g \in H(U, X): T(z)\, g(z)=0, z \in U\}, \qquad U \subset M$$

and the projection maps $KT(U) \rightarrow KT(V)$, where $V \subset U$, are restrictions. We want to show that KT is a *BACF* sheaf in the sense of Leiterer [4, Definition 2.1]. Fix $z_0 \in M$ and an arbitrary integer n. By Lemma 4.1(iv) there is a neighbourhood V_0 of z_0, a Banach space E_0 and a function $A_0 \in H(V_0, L(E_0, X))$ such that $\operatorname{Im} A_0(z)=\ker T(z)$ for $z \in V_0$. By

Lemma 1.9 $\limsup_{z \to z_0} k(A_0(z)) < +\infty$, and so also function $A_0(\cdot)$ satisfies all the equivalent conditions of Lemma 4.1, in particular (iv). Thus there exists a, possibly smaller, neighbourhood V_1 of z_0, a Banach space E_1, and a function $A_1 \in H(V_1, L(E_1, E_0))$ such that $\operatorname{Im} A_1(z) = \ker A_0(z)$ for $z \in V_1$ and (by Lemma 1.9) $\limsup_{z \to z_0} k(A_1(z)) < \infty$. We obtain by induction neighbourhoods $V_0, V_1, \ldots, V_n$ of z_0, Banach spaces $E_0, E_1, \ldots, E_n$ and operator-valued functions $A_i \in H(V_i L(E_i, E_{i-1}))$, $1 \leqslant i \leqslant n$, such that $\operatorname{Im} A_i(z) = \ker A_{i-1}(z)$ for $z \in V_i$. Let V be the intersection of $V_1, V_2, \ldots, V_n$. Consider the following sequence of sheaves of germs of holomorphic vector valued functions in V:

$$
{}_V\mathcal{O}^{E_n} \to {}_V\mathcal{O}^{E_{n-1}} \to \cdots \quad \cdots \to {}_V\mathcal{O}^{E_0} \to KT|_V \to 0
$$

(cf. [4] for the notation), with sheaf homomorphisms defined as operators of "multiplication" by operator valued functions $A_n(z)$, $A_{n-1}(z), \ldots, A_1(z), A_0(z)$ respectively. By Leiterer [4, Definition 2.1], in order to show that KT is $BACF$ it suffices to check that the sequence (4.4) is exact. We denote conveniently $E_{-1} = X$, $E_{-2} = Y$, and $A_{-1} = T$. If $z_1 \in V$, and g is a germ of holomorphic E_{i-1} valued function such that $A_{i-1} g = 0$, where $0 \leqslant i \leqslant n$, then the germ g can be represented by some function $g_1 \in A_+(P, E_{i-1})$, where P is a small compact polydisc centered at z_1 and contained in V. By Theorem 1 there is $h_1 \in A_+(P, E_i)$ such that $A_i(z) h_1(z) = g_1(z)$ for $z \in P$ and the germ h of h_1 at z_1 satisfies $A_i h = g$. Thus (4.4) is exact and so KT is $BACF$.

Applying Leiterer [4, Theorem 2.3(i) and Proposition 1.3] we get a Banach space E and a function $A \in H(M, L(E, X))$ such that the operator of multiplication by the function $A(z)$ induces a sheaf epimorphism. It remains to show that $\operatorname{Im} A(z) = \ker T(z)$ for every $z \in M$. Let $x_0 \in \ker T(z_0)$. Then by the Assertion in the proof of Lemma 4.1 there is a function $x \in H(W, X)$, where W is a neighbourhood of z_0, such that $T(z) x(z) = 0$ and $x(0) = x_0$. So the germ x at z_0 belongs to KT and there is a germ of holomorphic E-valued function f such that $Af = x$ as germs at z_0. It follows that $A(z_0) f(0) = x_0$. Thus $\operatorname{Im} A(z_0) = \ker T(z_0)$ an (iv) holds.

Having established the equivalence of (i)-(iv), we can apply it to the function $T(z)^*$.

(i) $\Rightarrow$ (vi) Also $T(z)^*$ satisfies (i), and applying (iv) to $T(z)^*$ we obtain a Banach space E and a function $C \in H(M, L(E, Y^*))$ such that $\operatorname{Im} C(z) = \ker T(z)^*$ for all $z \in M$. Set $B(z) = C(z)^* \mid Y$, with Y canonically embedded into Y^{**}. Denote $F = E^*$. Then $B \in H(M, L(Y, F))$. The argument (entirely local) that $\operatorname{Im} B(z)$ is closed and $\operatorname{Im} T(z) = \ker B(z)$ is the same as the one at the end of the proof of Lemma 4.1.

The implication (vi) $\Rightarrow$ (v) is obvious and (v) $\Rightarrow$ (iii) by Lemma 2.3. The proof is complete.

5. Applications: Banach Spaces of Sections of Parametrized Complexes of Quotient Spaces

In this section we apply results of Sections 3 and 5 to obtain a generalization of Corollary 3.5 in which the constant spaces $X^0, X^1, ..., X^m$ of the complex (3.4) are replaced by varying quotient spaces depending analytically on several parameters. This result (Theorem 5.1) will be applied in Slodkowski [6] to help represent some parts of the joint spectrum of several commuting operators as the graph of an analytic multifunction. It seems that Theorem 5.1, which in the present formulation looks rather technical, will become more natural if generalized to the context of analytic families of Banach spaces as defined in [7].

We need some additional notation and terminology to formulate the next theorem. Let P be a compact polydisc in $\mathbb{C}^k$ and G its open neighbourhood. If we are given operator-valued functions $T \in H(G, L(X, Y))$ and $S \in H(G, L(Y, Z))$, we denote

$$A_+(P, KS) = \{ g \in A_+(P, Y) \colon S(z)\, g(z) = 0,\ z \in P \},$$

$$A_+(P, IT) = \{ g \in A_+(P, Y) \colon \text{there is } f \in A_+(P, X) \text{ such that } g(z) = T(z)\, f(\ ),\ z \in P \}.$$

If $S(z) \circ T(z) = 0$ for $z \in P$, then $A_+(P, IT) \subset A_+(P, KS)$. In such case denote

$$A_+(P, KS/IT) = A_+(P, KS)/A_+(P, IT); \tag{5.1}$$

of course this defines a complex Banach space.

The next theorem deals with the following setting. The sets $P \subset G \subset \mathbb{C}^k$ are as above and let X_i, Y_i, Z_i, $i = 0, 1, ..., m$, are complex Banach spaces. Let functions $T_i \in H(G, L(X_i, Y_i))$, $S_i \in H(G, L(Y_i, Z_i))$, $i = 0, 1, ..., m$, and $\alpha_i \in H(G, L(X_i, X_{i+1}))$, $\beta_i \in H(G, L(Y_i, Y_{i+1}))$ and $\gamma_i \in H(G, L(Z_i, Z_{i+1}))$, $i = 0, 1, ..., m-1$, be given such that for each $z \in G$ each row and each column of the diagram

$$\begin{tikzcd}
0 \arrow[r] & Z_0 \arrow[r] & Z_1 \arrow[r] & \cdots & \cdots \arrow[r] & Z_{m-1} \arrow[r] & Z_m \arrow[r] & 0 \\
0 \arrow[r] & Y_0 \arrow[r] \arrow[u] & Y_1 \arrow[r] \arrow[u] & \cdots & \cdots \arrow[r] & Y_{m-1} \arrow[r] \arrow[u] & Y_m \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & X_0 \arrow[r] \arrow[u] & X_1 \arrow[r] \arrow[u] & \cdots & \cdots \arrow[r] & X_{m-1} \arrow[r] \arrow[u] & X_m \arrow[r] \arrow[u] & 0
\end{tikzcd} \tag{5.2}$$

is a cochain complex, provided the vertical differentials are defined by operators $T_i(z)$ and $S_i(z)$ (starting from the bottom in the column i) and the horizontal ones are defined by $\alpha_i(z)$, $\beta_i(z)$, and $\gamma_i(z)$.

THEOREM 5.1. *In addition to above denotations and conditions assume that the functions $z \to k(T(z))$ and $z \to k(S(z))$ are locally uniformly bounded on G and that the diagram* (5.2) *commutes. Then for every $z \in G$ the operators $\beta_0(z),\dots, \beta_{m-1}(z)$ induce differentials in the quotient complex*

$$\begin{aligned} 0 &\to \ker S_0(z)/\operatorname{Im} T_0(z) \to \ker S_1(z)/\operatorname{Im} T_1(z) \to \cdots \\ &\to \ker S_m(z)/\operatorname{Im} T_m(z) \to 0. \end{aligned}$$

Assume that this complex is exact for each $z \in G$. Then the complex,

$$0 \to A_+(P, KS_0/IT_0) \to A_+(P, KS_1/IT_1) \to \cdots \to A_+(P, KS_m/IT_m) \to 0,$$

with differentials naturally induced by operators $A_+(\beta_0)$, $A_+(\beta_1),\dots, A_+(\beta_{n-1})$, respectively, is exact as well.

Proof. Note that $A_+(\beta_i)$ induces the differential of the last complex because, by commutativity of diagram (5.1),

$$A_+(\beta_i)(A_+(P, KS_i)) \subset A_+(P, KS_{i+1})$$

and

$$A_+(\beta_i)(A_+(P, IT_i)) \subset A_+(P, IT_{i+1}).$$

For fixed i, $0 \leqslant i \leqslant m$, we will show exactness of the short sequence

$$A_+(P, KS_{i-1}/IT_{i-1}) \xrightarrow{\tilde{\delta}_{i-1}} A_+(P, KS_i/IT_i) \xrightarrow{\tilde{\delta}_i} A_+(P, KS_{i+1}/IT_{i+1}), \tag{5.3}$$

where $\tilde{\delta}_i$ denotes the differential induced by $A_+(\beta_i)$. We will determine first what does it mean that the short sequence

$$\begin{aligned} \ker S_{i-1}(z)/\operatorname{Im} T_{i-1}(z) &\xrightarrow{\delta_{i-1}(z)} \ker S_i(z)/\operatorname{Im} T_i(z) \\ &\xrightarrow{\delta_i(z)} \ker S_{i+1}(z)/\operatorname{Im} T_{i+1}(z) \end{aligned} \tag{5.4}$$

is exact, where $\delta_i(z)$ are the maps induced by $\beta_i(z)$. Observe that $[y] \in \ker \delta_i(z)$ if and only if $y \in \ker S_i(z)$ and $\beta_i(z) \in \operatorname{Im} T_{i+1}(z)$. On the other hand $[y] \in \operatorname{Im} \delta_{i-1}(z)$ if and only if there is $y' \in \ker S_{i-1}(z)$ such that $(y - \beta_{i-1}(y')) \in \operatorname{Im} T_i(z)$, that is $y \in \beta_{i-1}(z)(\ker S_{i-1}(z)) + \operatorname{Im} T_i(z)$. Thus the sequence (5.4) is exact if and only if

$$\ker S_i(z) \cap \beta_i(z)^{-1}(\operatorname{Im} T_{i+1}(z)) = \beta_{i-1}(z)(\ker S_{i-1}(z)) + \operatorname{Im} T_i(z). \tag{5.5}$$

Of course, the same argument shows that the complex (5.3) is exact if and only if

$$\ker A_+(S_i) \cap A_+(\beta_i)^{-1}(\operatorname{Im} A_+(T_{i+1})) \\ = A_+(\beta_{i-1})(\ker A_+(S_{i-1})) + \operatorname{Im} A_+(T_i). \tag{5.6}$$

We can assume without loss of generality that G is a Stein domain, e.g., an open polydisc. Since the functions $k(S_{i-1}(z))$ and $k(T_{i+1}(z))$ are locally bounded, we can apply Theorem 2 and obtain complex Banach spaces E and F and operator valued functions $A \in H(G, L(E, Y_{i-1}))$ and $B \in H(G, L(Y_{i+1}, F))$ such that

$$\ker S_{i-1}(z) = \operatorname{Im} A(z) \qquad \text{and} \qquad \operatorname{Im} T_{i+1}(z) = \ker B(z), \quad z \in G. \tag{5.7}$$

This representation of kernels by images and vice versa is the crucial step in our argument: it will make possible the deduction of Eq. (5.6) from Eqs. (5.5) by application of Theorem 1.

By (5.7), Eq. (5.5) can be rewritten as

$$\ker S_i(z) \cap \ker(B(z) \circ \beta_i(z)) = \operatorname{Im}(\beta_{i-1}(z) \circ A(z)) + \operatorname{Im} \mathrm{T}_i(z), \qquad z \in G. \tag{5.8}$$

Define $Q(z)(e \oplus x) = \beta_{i-1}(z)\, A(z)e + T_i(z)x$ for $e \in E$, $x \in X_i$ and $R(z)(y) = (S_i(z)y) \oplus (B(z)\, \beta_i(z)y)$, $y \in Y_i$. Then $Q \in H(G, L(E \oplus X_{i-1}, Y_i))$ and $R \in H(G, L(Y_i, Z_i \oplus F))$, and (5.8) reads

$$\ker R(z) = \operatorname{Im} Q(z) \qquad \text{for} \quad z \in G,$$

which, by Theorem 1 (cf. Remark 3.4) implies that

$$\ker A_+(R) = \operatorname{Im} A_+(Q), \tag{5.9}$$

where $A_+(Q)\colon A_+(P, E \oplus X_{i-1}) \to A_+(P, Y_i)$ and $A_+(R)\colon A_+(P, Y_i) \to A_+(P, Z_i \oplus F)$. In the same way Eqs. (5.7) and Theorem 1 imply that

$$\ker A_+(S_{i-1}) = \operatorname{Im} A_+(A) \qquad \text{and} \qquad \operatorname{Im} A_+(T_{i+1}) = \ker A_+(B);$$

due to these relations the Eq. (5.6) can be rewritten as

$$\ker A_+(S_i) \cap \ker A_+(B \circ \beta_i) = \operatorname{Im} A_+(\beta_{i-1} \circ A) + \operatorname{Im} A_+(T_i).$$

Since this equation is equivalent to (5.9), which has already been established, Eq. (5.6) holds. Q.E.D.

The next lemma, rather loosely connected with the main problems of this paper, will be needed in [6] to help implement the application of Theorem 5.1.

LEMMA 5.2. *Let P be a compact polydisc in $\mathbb{C}^k$, G an open neighbourhood of P, and $T \in H(G, L(X, Y))$, $S \in H(G, L(Y, Z))$. Assume that $S(z) \circ T(z) = 0$, $z \in G$, and that the functions $k(T(z))$ and $k(S(z))$ are locally bounded on G. Let $\tilde{Y} = A_+(P, KS/IT)$ be defined by (5.1). Denote by V_i the operator induced on $\tilde{Y}$ by operator of multiplication by z_i on $A_+(P, Y)$. Denote $Y_p := \operatorname{Im} V_1 + \cdots + \operatorname{Im} V_p$ for $0 < p \leqslant k$, and $Y_0 = (0)$. Then $\tilde{Y}_k$ is closed in $\tilde{Y}$ and V_p has zero kernel acting on $\tilde{Y}/\tilde{Y}_{p-1}$, for $p = 1, 2, ..., k$.*

Proof. Let $P = \bar{D}(0, r_1) \times \cdots \times \bar{D}(0, r_k)$ and denote $\bar{P}_p = D(0, r_{p+1}) \times \cdots \times \bar{D}(0, r_k)$ if $0 \leqslant p < k$.

Assertion 1. If $0 < p \leqslant k$, then

$$\tilde{Y}_p = \{[g] \in A_+(P, KS/IT): g(0, ..., 0, z_{p+1}, ..., z_k) = 0 \text{ for } |z_i| \leqslant r_i,\ p+1 \leqslant i \leqslant k\}.$$

Assertion 2. If $0 \leqslant p < k$, then the natural map from $\tilde{Y}/\tilde{Y}_p$ into $A_+(P_p, KS_p/IT_p)$, where $S_p(z_{p+1}, ..., z_k) = S(0, ..., 0, z_{p+1}, ..., z_k)$ and $T_p(z_{p+1}, ..., z_k) = T(0, ..., 0, z_{p+1}, ..., z_k)$ is an isomorphism onto.

First, observe that the lemma follows from the assertions. Indeed, by Assertion 1 the subspace Y_k is closed. Furthermore, under the identification of Assertion 2 operator V_{p+1} corresponds to the operator induced on $A_+(P_p, KS_p/IT_p)$ by multiplication by z_{p+1}. To check that this operator is one-to-one we let $[g] \in A_+(P_p, KT_p/IT_p)$ be such that $[z_{p+1} g] = 0$, i.e., $z_{p+1} g(z_{p+1}, ..., z_k)$ belongs to $\operatorname{Im} T_p(z_{p+1}, ..., z_k)$ for $|z_i| \leqslant r_i$, $i = p+1, ..., k$. Therefore $g(z'') \in \operatorname{Im} T_p(z'')$, whenever $z'' \in P_p$ and $z''_{p+1} \neq 0$, and so by Theorem 2(ii) $g(z'') \in \operatorname{Im} T_p(z'')$ for all $z'' \in P_p$. By Theorem 1 $g \in \operatorname{Im} A_+(T_p)$. Thus $[g] = 0$, as required. It remains to check the assertions.

The inclusion $(\subset)$ in Assortion 1 is obvious. For the reverse inclusion consider g in $A_+(P, Y)$ such that $S(z) g(z) = 0$, $z \in P$ and $g(z) = 0$ if $z_1 = \cdots = z_p = 0$, $z \in P$. By Theorem 2(iv) applied twice there are Banach spaces E_0, E_1 and functions $A_0 \in H(G, L(E_0, Y))$, $A_1 \in H(G, L(E_1, E_0))$, such that $\operatorname{Im} A_1(z) = \ker A_0(z)$ and $\operatorname{Im} A_0(z) = \ker S(z)$, $z \in G$ (we can assume that G is Stein). By Theorem 1 there is $h \in A_+(P, E_0)$ such that $g(z) = A_0(z) h(z)$, $z \in P$. In the same way there is $h_1 \in A_+(P_p, E_1)$ such that $A_1(z'') h_1(z'') = h_0(0, z'')$ for $z'' \in P_p$. Define $h_2(z) = h_1(0, ..., 0, z_{p+1}, ..., z_k)$ for $z \in P$. Of course $h_2 \in A_+(P, E_1)$. Set $h = h_0 - A_+(A_1)h_2$. Then $h \in A_+(P, E_0)$ and $h(z) = 0$ whenever $z_1 = \cdots = z_p = 0$. A standard argument shows that there are functions $h_1, ..., h_p \in A_+(P, E_0)$ such that $h(z) = z_1 h_1(z) + \cdots + z_p h_p(z)$, $z \in P$. Set $g(z) = A_0(z) h_i(z)$, $z \in P$. Then $g_i \in A_+(P, Y)$ and $S(z) g_i(z) = 0$ for $1 \leqslant i \leqslant p$, and so $[g_i] \in \tilde{Y}$. Since $g(z) = A_0(z) h_0(z) = A_0(z) h(z)$, $z \in P$, we conclude that $[g] = [z_1 g_1] + \cdots + [z_p g_p]$, i.e., $[g] \in \tilde{Y}_p$, as required.

Let $\varphi: \tilde{Y}_p \to A_+(P_p, KS_p/IT_p)$ denote the map in Assertion 2. To see that it is one-to-one, let $g \in A_+(P, KS)$ be such that $\varphi([g]) = [0]$, which means that $g(z) \in \operatorname{Im} T(z)$, whenever $z_1 = \cdots = z_p = 0$, $z \in P$. By Theorem 1 (and Theorem 2) there exists $h_p \in A_+(P, X)$ such that $g(0, z'') = T(0, z'')\, h_p(z'')$, for $z'' \in P_p$. Set $h(z) = h_p(0, z'')$ for $z = (z_1, ..., z_p, z'') \in P$. Then $h \in A_+(P, X)$ and if we set $g_1 = g - A_+(T)h$, then $g_1 \in A_+(P, KS)$ and moreover $g_1(0, ..., 0, z_{p+1}, ..., z_k) = 0$, $(z_{p+1}, ..., z_k) \in P_p$. By Assertion 1 $[g_1] \in \tilde{Y}_p$ and since $g_1 - g \in \operatorname{Im} A_+(T)$, $[g] \in \tilde{Y}_p$, and so φ is one-to-one.

To show that φ is onto we obtain, by Theorem 2(iv), a Banach space E and a function $A \in H(G, L(E, Y))$ such that $\operatorname{Im} A(z) = \ker S(z)$, $z \in G$. Let $g_p \in A_+(P_p, KS_p)$. By Theorem 1 there exists $f_p \in A_+(P_p, E)$ such that $g_p(z'') = A(0, z'')\, f_p(z'')$, $z'' \in P_p$. Set $f(z) = f_p(z'')$ for $z = (z_1, ..., z_p, z'') \in P$, and $g(z) = A(z)\, f(z)$ for $z \in P$. Then $g \in A_+(P, KS)$ and $\varphi([g] + \tilde{Y}_p) = [g_p]$. Q.E.D.

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